Computational Cognitive Modeling

Probabilistic programming, program induction, and language of thought models

Brenden Lake & Todd Gureckis

email address for instructors:
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course website:
https://brendenlake.github.io/CCM-site/
class world():
    def __init__(self):
        self.dict_strength = {}
    def clear(self): # used when sampling over possible world
        self.dict_strength = {}

    def strength(name):
        if name not in W.dict_strength:
            W.dict_strength[name] = 0

    def lazy(name):
        return random.random() < 0.1

    def team_strength(team):
        # team : list of names
        mysum = 0.
        for name in team:
            if lazy(name):
                mysum += (strength(name) / 2.)
            else:
                mysum += strength(name)
        return mysum

    def winner(team1,team2):
        # team1 : list of names
        # team2 : list of names
        if team_strength(team1) > team_strength(team2):
            return team1
        else:
            return team2

    def beat(team1,team2):
        return winner(team1,team2) == team1

(Example from homework; Goodman et al., 2015)
Stochastic program (known structure)

Data
(e.g., game matches and who won)

class world():
def __init__(self):
    self.dict_strength = {}
def clear(self): # used when sampling over possible world
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def strength(name):
    if name not in W.dict_strength:

def lazy(name):
    return random.random() < 0.1

How strong is Bob?
Probabilistic programs / probabilistic programming

- Probabilistic program: A probabilistic model defined in a structured description language (much like a programming language) using random programming primitives.

- Due to random primitives, every time the program executes it returns a different output.

- Probabilistic programs are a generalization of Bayesian networks, and many of the other Bayesian models we have discussed.

- Especially convenient when the prior is too complex to write down as a set of hypotheses, or the model is awkward or impossible to write as a Bayesian network.
Probabilistic programs: A simple example

Preliminary definitions

```python
def flip(theta=0.5):
    return random.random() < theta
```

Simple probabilistic program

```
A = flip()
B = flip()
C = flip()
D = A + B + C
```

Bayesian inference

```
P(D)   P(A|D = 3)   P(A|D ≥ 2)
```

(again, notice productivity reasoning)

Example from Noah Goodman and Josh Tenenbaum

https://probmods.org/
Probabilistic program or Bayesian network?

A = flip()
B = flip()
C = flip()
D = A + B + C

In this case, the probabilistic program can be straightforwardly represented as a Bayesian network, although the program representation conveys more information.
Probabilistic programs: Another example

Simple probabilistic program (yet more complex than before)

\[
A = \text{flip()}
\]
\[
B = \text{flip()}
\]
\[
C = \text{flip()}
\]
\[
\text{if } C:
\]
\[
D = A + B + C
\]
\[
\text{else:}
\]
\[
E = \text{flip()}
\]
\[
F = (2*\text{flip()})^2
\]
\[
D = A + B + C + E + F
\]

Bayesian inference

\[
P(D)
\]
\[
P(A|D \geq 2)
\]
A = flip()
B = flip()
C = flip()

if C:
    D = A + B + C
else:
    E = flip()
    F = (2*flip())**2
    D = A + B + C + E + F

Bayesian networks (graphical models) do not have a mechanism for adding additional variables, and they lack general control structures that are relevant in both cognitive science and data science applications (if statements, for loops, while loops, recursion, etc.)
class world():
    def __init__(self):
        self.dict_strength = {}
    def clear(self):  # used when sampling over possible world
        self.dict_strength = {}

    def strength(name):
        if name not in W.dict_strength:

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        else:
            return team2

    def beat(team1,team2):
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(see Goodman et al., 2015)
Reasoning about tennis with probabilistic programs

<table>
<thead>
<tr>
<th>confounded evidence</th>
<th>strong indirect evidence</th>
<th>weak indirect evidence</th>
<th>diverse evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>(3,4)</td>
<td>(5,6)</td>
<td>(7,8)</td>
</tr>
<tr>
<td>A &gt; B</td>
<td>A &gt; B</td>
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</tr>
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</table>

Human judgements

Model judgements

(see Gerstenberg et al., 2012; Goodman et al., 2015)
Bayesian network formulation depends on set of players

Each specification of the teams changes the Bayes net structure, and thus an additional modeling mechanism is needed to construct the Bayes nets.

Sample anew for each game.
Example probabilistic programming languages and software

- WebPPL
- pyro
- Edward
- Church
- Stan
Prior distribution over (unknown) possible programs

Hypothesis 1
\[ \lambda S . (\text{if } (\text{singleton？} S) \text{ “one”} \text{ if }\ (\text{doubleton?} S) \text{ “two”} \text{ undefined})) \]

Hypothesis 2
\[ \lambda S . (\text{if } (\text{singleton?} S) \text{ “one”} \text{ if } (\text{doubleton?} S) \text{ “two”} \text{ undefined})) \]

Hypothesis 3
\[ \lambda S . (\text{if } (\text{singleton?} S) \text{ “one”} \text{ if }\ (\text{doubleton?} S) \text{ “two”} \text{ undefined})) \]

Hypothesis N ...

Data (D)

Which is the right program?

Most likely to have generated the data?
Program induction

- Data is generated from an unknown program, where unlike standard probabilistic programming, we don’t know the structure of the program.

- Prior over programs is usually defined by assuming a set of programming primitives and combination operations, which is also referred to as a “Language of thought” model in cognitive science (a la Jerry Fodor).

- More analogous to “structure learning” for Bayesian networks, where we are searching for the right causal model that generated the data.
Language of thought / program induction in Python

**LOTlib**

LOTlib is a Python 2 library for implementing "language of thought" models. A LOTlib model specifies a set of primitives and captures learning as inference over compositions of those primitives in order to express complex concepts. LOTlib permits lambda expressions, meaning that learners can come up with abstractions over compositions and define new...
Motivation: We need more than Bayesian networks to represent complex, real causal processes for generating data

same causal process

\[ \begin{array}{c}
\text{2} \\
\text{Pencil}
\end{array} \]

\[ \rightarrow \]

different examples

\[ \begin{array}{ccc}
\text{2} & \text{2} & \text{2} \\
\text{2} & \text{Z} & \text{2} \\
\text{2} & \text{2} & \text{2}
\end{array} \]

(Figure credit: Hinton & Nair, 2006)

Is it growing too close to my house?

How will it grow if I trim it?

state-of-the-art neural net caption generation:
“A group of people standing on top of a beach”
Human-level concept learning through probabilistic program induction

Brenden M. Lake, Ruslan Salakhutdinov, Joshua B. Tenenbaum

People learning new concepts can often generalize successfully from just a single example, yet machine learning algorithms typically require tens or hundreds of examples to perform with similar accuracy. People can also use learned concepts in richer ways than conventional algorithms—for action, imagination, and explanation. We present a computational model that captures these human learning abilities for a large class of simple visual concepts: handwritten characters from the world’s alphabets. The model represents concepts as simple programs that best explain observed examples under a Bayesian criterion. On a challenging one-shot classification task, the model achieves human-level performance while outperforming recent deep learning approaches. We also present several “visual Turing tests” probing the model’s creative generalization abilities, which in many cases are indistinguishable from human behavior.

Despite remarkable advances in artificial intelligence and machine learning, two aspects of human conceptual knowledge have eluded machine systems. First, for most interesting kinds of natural and man-made categories, people can learn a new concept from just one or a handful of examples, whereas standard algorithms in machine learning require tens or hundreds of examples to perform similarly. For instance, people may only need to see one example of a novel two-wheeled vehicle (Fig. 1A) in order to grasp the boundaries of the new concept, and even children can make meaningful generalizations via “one-shot learning” (1–3). In contrast, many of the leading approaches in machine learning are also the most data-hungry, especially “deep learning” models that have achieved new levels of performance on object and speech recognition benchmarks (4–9). Second, people learn richer representations than machines do, even for simple concepts (Fig. 1B), using them for a wider range of functions, including (Fig. 1, ii) creating new exemplars (10), (Fig. 1, iii) parsing objects into parts and relations (11), and (Fig. 1, iv) creating new abstract categories of objects based on existing categories (12, 13). In contrast, the best machine classifiers do not perform these additional functions, which are rarely studied and usually require specialized algorithms. A central challenge is to explain these two aspects of human-level concept learning: How do people learn new concepts from just one or a few examples? And how do people learn such abstract, rich, and flexible representations? An even greater challenge arises when putting them together: How can learning succeed from such sparse data yet also produce such rich representations? For any theory of

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Standard machine learning approach: Deep neural network with large amounts of data

MNIST:
10 classes of handwritten digits
6,000 examples each

Layer 1: 4 feature maps
Filter bank + non-linearity
Max pooling

Layer 2: 12 feature maps
Filter bank + non-linearity
Max pooling

Output classes
0 1 2... 9

Slide credit: Yann LeCun
People can learn a new concept from a single image.

Less data

People can apply their knowledge flexibly to new tasks.

More knowledge

Learning a new letter: Comparing humans and machines

Generating new concepts

Generating new examples

Parsing
People can learn a new concept from a single image.

People can apply their knowledge flexibly to new tasks.
Omniglot stimulus set
(https://github.com/brendenlake/omniglot)

1600+ concepts
20 examples each
probabilistic motor programs

People drawing a new character

Wednesday, October 17, 2012

Human drawings
The number of subparts $n_i$ for each part $i = 1, \ldots, k$, from their empirical distributions as measured from the background set. Second, a template for each part $S_i$ is constructed by sampling subparts from a set of discrete primitive actions learned from the background set (Fig. 3A, i), such that the probability of the next action depends on the previous. Third, parts are then grounded as parameterized curves (splines) by sampling the control points and scale parameters for each subpart. Last, parts are roughly positioned to begin either independently, at the beginning, or together, at the beginning.

Fig. 3. A generative model of handwritten characters. (A) New types are generated by choosing primitive actions (color coded) from a library (i), combining these subparts (ii) to make parts (iii), and combining parts with relations to define simple programs (iv). New tokens are generated by running these programs (v), which are then rendered as raw data (vi). (B) Pseudocode for generating new types $y$ and new token images $I(m)$ for $m = 1, \ldots, M$. The function $f(\cdot, \cdot)$ transforms a subpart sequence and start location into a trajectory.

Human parses

Machine parses

Training item with model's five best parses

Test items

Human drawings

stroke order: 1 2 3 4 5
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The function $f(·, ·)$ transforms a subpart sequence and start location into a trajectory.

Human parses

Machine parses

Training item with model's five best parses

Test items

stroke order:

Human drawings

- Original

- Human drawings

- Stroke order:
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\[ \text{Fig. 3. A generative model of handwritten characters.} \]

\[ \text{(A) New types are generated by choosing primitive actions (color coded) from a library (i), combining these subparts (ii) to make parts (iii), and combining parts with relations to define simple programs (iv). New tokens are generated by running these programs (v), which are then rendered as raw data (vi).} \]

\[ \text{B) Pseudocode for generating new types and new token images $I_m$ for} \]

\[ m = 1, \ldots, M. \]

\[ \text{The function $f(\cdot, \cdot)$ transforms a subpart sequence and start location into a trajectory.} \]
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The function \( f(\cdot, \cdot) \) transforms a subpart sequence and start location into a trajectory.

**Fig. 3.** A generative model of handwritten characters.

(A) New types are generated by choosing primitive actions (color coded) from a library (i), combining these subparts (ii) to make parts (iii), and combining parts with relations to define simple programs (iv). New tokens are generated by running these programs (v), which are then rendered as raw data (vi).

(B) Pseudocode for generating new types and new token images \( I(m) \) for \( m = 1, \ldots, M \).

The function \( f(\cdot, \cdot) \) transforms a subpart sequence and start location into a trajectory.

**Human parses**

**Machine parses**

**Human drawings**

<table>
<thead>
<tr>
<th>Human drawings</th>
<th>Machine parses</th>
<th>Human parses</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Human drawings" /></td>
<td><img src="image2.png" alt="Machine parses" /></td>
<td><img src="image3.png" alt="Human parses" /></td>
</tr>
</tbody>
</table>

**Fig. 4.** Inferring motor programs from images.

Parts are distinguished by color, with a colored dot indicating the beginning of a stroke and an arrowhead indicating the end. (A) The top row shows the five best programs discovered for an image along with their log-probability scores (Eq. 1). For classification, each program was refit to three new test images (left in image triplets), and the best-fitting parse (top right) is shown with its image reconstruction (bottom right) and classification score (log posterior predictive probability). Subpart breaks are shown as black dots.

(B) Nine human drawings of three characters (left) are shown with their ground truth parses (middle) and best model parses (right).

stroke order: 1 2 3 4 5
the number of subparts \( n_i \), for each part \( i = 1, \ldots, k \), from their empirical distributions as measured from the background set. Second, a template for a part \( S_i \) is constructed by sampling subparts from a set of discrete primitive actions learned from the background set (Fig. 3A, i), such that the probability of the next action depends on the previous. Third, parts are then grounded as parameterized curves (splines) by sampling the control points and scale parameters for each subpart. Last, parts are roughly positioned to begin either independently, at the beginning, SCIENCE sciencemag.org 00 MONTH 2015 • VOL 000 ISSUE 0000

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The function \( f(\cdot, \cdot) \) transforms a subpart sequence and start location into a trajectory.

The function \( f(\cdot, \cdot) \) transforms a subpart sequence and start location into a trajectory.
Human parses

Machine parses

Human drawings

stroke order: 1 2 3 4 5
Probabilistic program induction model of concept learning

- **primitives** (1D curvelets, 2D patches, 3D geons, actions, sounds, etc.)
- **sub-parts**
- **parts**
- **object template**
- **type level**
- **token level**
- **exemplars**
- **raw data**

Bayes’ rule

\[
P(\theta|I) = \frac{P(I|\theta)P(\theta)}{P(I)}
\]

- **latent program**
- **raw binary image**
- **renderer**
- **prior on parts, relations, etc.**
Task: “Generate a new example”
A “visual Turing test” for generating new examples
A “visual Turing test” for generating new examples
A "visual Turing test" for generating new examples

machine generated
A “visual Turing test” for generating new examples

machine generated
Task: “Generate a new character from the same alphabet”
A “visual Turing test” for generating new concepts
A “visual Turing test” for generating new concepts
A “visual Turing test” for generating new concepts

machine generated
A “visual Turing test” for generating new concepts

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>![A1]</td>
<td>![B1]</td>
</tr>
<tr>
<td>![A2]</td>
<td>![B2]</td>
</tr>
<tr>
<td>![A3]</td>
<td>![B3]</td>
</tr>
</tbody>
</table>

machine generated
Probabilistic program induction

- Primitives: 1D curvelets, 2D patches, 3D geons, actions, sounds, etc.
- Sub-parts
- Parts
- Object template
- Type level
- Token level
- Exemplars
- Raw data

Bayes' rule:
\[
P(\theta|I) = \frac{P(I|\theta)P(\theta)}{P(I)}
\]
primitives
(1D curvelets, 2D patches, 3D geons, actions, sounds, etc.)

sub-parts

parts

object template

type level

---

**Probabilistic program induction**

procedure **GENERATE_TYPE**

\[
\begin{align*}
\kappa &\leftarrow P(\kappa) \\
\text{for } i = 1 \ldots \kappa \text{ do} &
\begin{align*}
\quad n_i &\leftarrow P(n_i|\kappa) \\
\quad \text{for } j = 1 \ldots n_i \text{ do} &
\begin{align*}
\quad s_{ij} &\leftarrow P(s_{ij}|s_{i(j-1)}) \\
\quad \text{end for}
\end{align*}
\end{align*}
\end{align*}
\]

\begin{itemize}
\item \(\text{Sample number of parts}\)
\item \(\text{Sample number of sub-parts}\)
\item \(\text{Sample sub-part sequence}\)
\item \(\text{Sample relation}\)
\end{itemize}

\[
\begin{align*}
R_i &\leftarrow P(R_i|S_1, \ldots, S_{i-1}) \\
\psi &\leftarrow \{\kappa, R, S\} \\
\text{return } &\@\text{GENERATE_TOKEN}(\psi)
\end{align*}
\]

\(\text{Return program}\)
primitives
(1D curvelets, 2D patches, 3D geons, actions, sounds, etc.)

sub-parts

parts

relation: attached along

object template

procedure \textsc{GenerateToken}(\psi)

\begin{align*}
\text{for } i = 1 \ldots \kappa \text{ do} & \\
S_i^{(m)} & \leftarrow P(S_i^{(m)} | S_i) \quad \triangleright \text{Add motor variance} \\
L_i^{(m)} & \leftarrow P(L_i^{(m)} | R_i, T_i^{(m)}, \ldots, T_{i-1}^{(m)}) \quad \triangleright \text{Sample part's start location} \\
T_i^{(m)} & \leftarrow f(L_i^{(m)}, S_i^{(m)}) \quad \triangleright \text{Compose a part's trajectory} \\
\text{end for} \\
A^{(m)} & \leftarrow P(A^{(m)}) \quad \triangleright \text{Sample affine transform} \\
I^{(m)} & \leftarrow P(I^{(m)} | T^{(m)}, A^{(m)}) \quad \triangleright \text{Sample image} \\
\text{return } I^{(m)}
\end{align*}

type level

token level

exemplars

raw data
Learning a prior distribution over programs

learned action primitives

learned primitive transitions

1250 primitives
scale selective
translation invariant
Learning a prior distribution over programs

number of strokes

number of sub-strokes for a character with $\kappa$ strokes

global transformations

relations between strokes

independent (34%)  attached at start (5%)  attached at end (11%)  attached along (50%)
Approximate probabilistic inference

\[ P(\theta|I) = \frac{P(I|\theta)P(\theta)}{P(I)} \]

such that

\[ w_i \propto P(\theta[i]|I) \]

Intuition: Fit strokes to the observed pixels as closely as possible, with these constraints:
- fewer strokes
- high-probability primitive sequence
- use relations
- stroke order
- stroke directions
Approximate probabilistic inference

Step 1: characters as undirected graphs

Step 2: guided random parses

Step 3: Top-down fitting with gradient-based optimization

log-probability

more likely

less likely

a)

b)
Human-level concept learning

the speed of learning

the richness of representation

parsing

generating new concepts

generating new examples
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Machine parses

Human drawings

stroke order: 1 2 3 4 5

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Machine parses

Test items

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<td>Human drawings</td>
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</tbody>
</table>

Training item with model's five best parses

Test items

Fig. 4. Inferring motor programs from images. Parts are distinguished by color, with a colored dot indicating the beginning of a stroke and an arrowhead indicating the end. (A) The top row shows the five best programs discovered for an image along with their log-probability scores (Eq. 1). For classification, each program was refit to three new test images (left in image triplets), and the best-fitting parse (top right) is shown with its image reconstruction (bottom right) and classification score (log posterior predictive probability). Subpart breaks are shown as black dots. (B) Nine human drawings of three characters (left) are shown with their ground truth parses (middle) and best model parses (right).
the number of subparts \( n_i \), for each part \( i = 1, \ldots, k \), from their empirical distributions as measured from the background set. Second, a template for a part \( S_i \) is constructed by sampling subparts from a set of discrete primitive actions learned from the background set (Fig. 3A, i), such that the probability of the next action depends on the previous. Third, parts are then grounded as parameterized curves (splines) by sampling the control points and scale parameters for each subpart. Last, parts are roughly positioned to begin either independently, at the beginning, or at the end of a line, or both (Fig. 3A, ii).
the number of subparts

from their empirical distributions as measured

Second, a template for a part $S_i$ is constructed by sampling subparts from a set of discrete primitive actions learned from the background set (Fig. 3A, i), such that the probability of the next action depends on

the previous.

Third, parts are then grounded as parameterized curves (splines) by sampling the control points and scale parameters for each subpart. Last, parts are roughly positioned to begin either independently, at the beginning,

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VOL 000 ISSUE 0000

Fig. 3. A generative model of handwritten characters.

(A) New types are generated by choosing primitive actions (color coded) from a library (i), combining these subparts (ii) to make parts (iii), and combining parts with relations to define simple programs (iv). New tokens are generated by running these programs (v), which are then rendered as raw data (vi).

(B) Pseudocode for generating new types $y$ and new token images $I(m)$ for $m = 1, \ldots, M$.

The function $f(\cdot, \cdot)$ transforms a subpart sequence and start location into a trajectory.

Human parses

Machine parses

Human drawings

<p>| | | |</p>
<table>
<thead>
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</tr>
</tbody>
</table>

stroke order: 1 2 3 4 5

Human parses

Machine parses

Human drawings

<p>| | | |</p>
<table>
<thead>
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Test items

Parts are distinguished by color, with a colored dot indicating the beginning of a stroke and an arrowhead indicating the end. (A) The top row shows the five best programs discovered for an image along with their log-probability scores (Eq. 1). For classification, each program was refit to three new test images (left in image triplets), and the best-fitting parse (top right) is shown with its image reconstruction (bottom right) and classification score (log posterior predictive probability). Subpart breaks are shown as black dots.

(B) Nine human drawings of three characters (left) are shown with their ground truth parses (middle) and best model parses (right).
Human-level concept learning

the speed of learning

the richness of representation

parsing

generating new concepts

generating new examples
One-shot classification

Training item with model’s five best parses

Test items

fits

reconstruction

\[ P(I_{\text{new}} | I_{\text{old}}) \]
Do people represent static characters by their causal dynamics?

Behavioral evidence

- Writing experience influences perception
  (Freyd, 1983; Tse & Cavanagh, 2000; Knoblich & Prinz, 2001; James & Gauthier, 2009).

- Inferring the dynamics from static letters.
  (Babcock & Freyd, 1988)

Neuroimaging evidence

- Writing experience changes the functional specialization of visual cortex for letters.
  (James & Atwood, 2009; James, 2010)

- Motor areas of cortex respond to static letters.
  (Anderson et al., 1990; Loncamp et al., 2003; James & Gauthier, 2006; Longcamp et al., 2006; Longcamp et al., 2010)
One-shot classification performance

After all models pre-trained on 30 alphabets of characters.

Program induction models

People
BPL
BPL Lesion (wrong prior)
BPL Lesion (no compositionality)

Deep neural networks
(no causality)
Deep Siamese Convnet
(Koch, Zemel, Salakhutdinov. 2015)
Deep Convnet
Hierarchical Deep

Error rate (%)
Human-level concept learning

the speed of learning

the richness of representation

parsing

generating new concepts

generating new examples
A “visual Turing test” for generating new concepts
A “visual Turing test” for generating new concepts
More large-scale behavioral experiments

Generating new examples (dynamic)

Generating new concepts (from type)

Generating new concepts (unconstrained)

59% correct in visual Turing test
6 of 30 judges above chance

49% correct in visual Turing test
8 of 35 judges above chance

51% correct in visual Turing test
2 of 25 judges above chance
More large-scale behavioral experiments

Generating new examples (dynamic)

Generating new concepts (from type)

Human or Machine?

Generating new concepts (unconstrained)

Human or Machine?

59% correct in visual Turing test
6 of 30 judges above chance

49% correct in visual Turing test
8 of 35 judges above chance

51% correct in visual Turing test
2 of 25 judges above chance
In acquiring number words, children exhibit a qualitative leap in which they transition from understanding a few number words, to possessing a rich system of interrelated numerical concepts. We present a computational framework for understanding this inductive leap as the consequence of statistical inference over a sufficiently powerful representational system. We provide an implemented model that is powerful enough to learn number word meanings and other related conceptual systems from naturalistic data. The model shows that bootstrapping can be made computationally and philosophically well-founded as a theory of number learning. Our approach demonstrates how learners may combine core cognitive operations to build sophisticated representations during the course of development, and how this process explains observed developmental patterns in number word learning.
Children’s development of numerical concepts

“Give-a-number” task
“Give me two”
“Give me three”

(Wynn, 1990; Wynn, 1992)
Children’s development of numerical concepts

Children progress through a series of stages
• “one-knower”, “two-knower,” “three-known,” “four-knower” (sometimes), and then “cardinal-principle knower”

Example: “two knower”

give me one: 🍏

give me two: 🍏 🍏

give me three: 🍏 🍏 🍏 OR 🍏 🍏 OR 🍏 ...

(inconsistent; arbitrary response beyond “two”)


Children’s development of numerical concepts

- Critically, children can count well-beyond the range of their “knower” status, yet they don’t understand the meaning of the numbers.
- Transition from “N-knower” to “CP-knower” happens roughly between ages 2.5 and 3.5

Patterns of success in give-a-number task in Experiment 3

<table>
<thead>
<tr>
<th>Success pattern</th>
<th>Number of children</th>
<th>Mean age</th>
<th>Counting ability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(range)</td>
</tr>
<tr>
<td>one-knower</td>
<td>- - - - - -</td>
<td>1</td>
<td>2:8</td>
</tr>
<tr>
<td>two-knower</td>
<td>+ - - - - -</td>
<td>3</td>
<td>3:0</td>
</tr>
<tr>
<td>three-knower</td>
<td>+ + - - - -</td>
<td>2</td>
<td>2:11</td>
</tr>
<tr>
<td>CP-knower</td>
<td>+ + + + + +</td>
<td>7</td>
<td>3:7</td>
</tr>
</tbody>
</table>

Note: “+” indicates success on a numerosity; “-” indicates failure.

(data from Wynn, 1990)
Programming primitives allowed in the language of thought model

<table>
<thead>
<tr>
<th>Functions mapping sets to truth values</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(singleton? X)</td>
<td>Returns true iff the set X has exactly one element</td>
</tr>
<tr>
<td>(doubleton? X)</td>
<td>Returns true iff the set X has exactly two elements</td>
</tr>
<tr>
<td>(tripleton? X)</td>
<td>Returns true iff the set X has exactly three elements</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Functions on sets</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(set-difference X Y)</td>
<td>Returns the set that results from removing Y from X</td>
</tr>
<tr>
<td>(union X Y)</td>
<td>Returns the union of sets X and Y</td>
</tr>
<tr>
<td>(intersection X Y)</td>
<td>Returns the intersect of sets X and Y</td>
</tr>
<tr>
<td>(select X)</td>
<td>Returns a set containing a single element from X</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Logical functions</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(and P Q)</td>
<td>Returns TRUE if P and Q are both true</td>
</tr>
<tr>
<td>(or P Q)</td>
<td>Returns TRUE if either P or Q is true</td>
</tr>
<tr>
<td>(not P)</td>
<td>Returns TRUE iff P is false</td>
</tr>
<tr>
<td>(if P X Y)</td>
<td>Returns X iff P is true, Y otherwise</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Functions on the counting routine</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(next W)</td>
<td>Returns the word after W in the counting routine</td>
</tr>
<tr>
<td>(prev W)</td>
<td>Returns the word before W in the counting routine</td>
</tr>
<tr>
<td>(equal-word? W V)</td>
<td>Returns TRUE if W and V are the same word</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Recursion</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L S)</td>
<td>Returns the result of evaluating the entire current lambda expression on set L with S as a parameter</td>
</tr>
</tbody>
</table>

(Piantadosi, Tenenbaum & Goodman)
### Example hypotheses in a language of thought

**One-knower**

\[ \lambda S . \begin{cases} \text{“one”} & \text{if (singleton? } S) \\ \text{undef} & \text{otherwise} \end{cases} \]

**Two-knower**

\[ \lambda S . \begin{cases} \text{“one”} & \text{if (singleton? } S) \\ \text{“two”} & \text{if (doubleton? } S) \\ \text{undef} & \text{otherwise} \end{cases} \]

**Three-knower**

\[ \lambda S . \begin{cases} \text{“one”} & \text{if (singleton? } S) \\ \text{“two”} & \text{if (doubleton? } S) \\ \text{“three”} & \text{if (tripleton? } S) \\ \text{undef} & \text{otherwise} \end{cases} \]

**CP-knower**

\[ \lambda S . \begin{cases} \text{“one”} & \text{if (singleton? } S) \\ \text{“two”} & \text{if (doubleton? } S) \\ \text{“three”} & \text{if (tripleton? } S) \\ \text{“four”} & \text{if (quadrotion? } S) \\ \text{“five”} & \text{otherwise} \end{cases} \]

**Mod-5**

\[ \lambda S . \begin{cases} \text{“one”} & \text{if (or (singleton? } S) \\ \text{“five”} & \text{otherwise} \end{cases} \]

\[ \begin{cases} \text{“one”} & \text{if (or (singleton? } S) \\ \text{“five”} & \text{otherwise} \end{cases} \]

### Example set S

![Example set S](example-set.png)

\( \lambda S \). indicates a function that takes a set \( S \) as an argument.
Defining a prior distribution over programs
(a “Probabilistic Language of Thought”)

Formalism used:
probabilistic context-free grammar

Boolean

(if B the W else W)

(if (singleton? S) then W else W)

(if (singleton? S) then “one” else W)

(if (singleton? S) then “one” else “undefined”)

Word

W

(if (and B B) W else W)

(if (and B B) W else W)

(if (and (True) B) W else W)

...
Probabilistic model over programs

Example $L$

$$\lambda S . \begin{cases} \text{if (singleton? } S) \quad \text{"one"} \\ \text{undef} \end{cases}$$

Probabilistic model

$$P(L)$$  

prior on programs $L$ (defined with probabilistic grammar)

$$P(D|L)$$  

Noisy likelihood where right number is usually returned, but with some noise

Bayes’ rule for learning programs

$$P(L|D) = \frac{P(D|L)P(L)}{P(D)}$$
Learning as program induction

Program ($L$)  $\lambda S . (if \ (singleton? \ S) \ "one" \ (next \ (L \ (set-difference \ S \ (select \ S))))))$

$$P(L|D) = \frac{P(D|L)P(L)}{P(D)}$$
Results: Program induction model follows a similar developmental trajectory

<table>
<thead>
<tr>
<th>Amount of data</th>
<th>Posterior probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>50</td>
<td>0.8</td>
</tr>
<tr>
<td>100</td>
<td>0.6</td>
</tr>
<tr>
<td>150</td>
<td>0.4</td>
</tr>
<tr>
<td>200</td>
<td>0.2</td>
</tr>
<tr>
<td>250</td>
<td>0.0</td>
</tr>
<tr>
<td>300</td>
<td>0.0</td>
</tr>
<tr>
<td>350</td>
<td>0.0</td>
</tr>
</tbody>
</table>

One-knower:
\[
\lambda S. (\text{if (singleton? } S) \text{ "one" } \text{ undefined})
\]

Two-knower:
\[
\lambda S. (\text{if (singleton? } S) \text{ "one" } \text{if (doubleton? } S) \text{ "two" undefined})
\]

Three-knower:
\[
\lambda S. (\text{if (singleton? } S) \text{ "one" } \text{if (doubleton? } S) \text{ "two" } \text{if (tripleton? } S) \text{ "three" undefined})
\]

CP-knower:
\[
\lambda S. (\text{if (singleton? } S) \text{ "one" } \text{if (doubleton? } S) \text{ "two" } \text{if (tripleton? } S) \text{ "three" undefined})
\]

Graph shows the marginal posterior probability of exhibiting each type of behavior, as a function of amount of data and how it relates to the model's predicted distribution of responses.
Case study: Learning by asking questions

Question Asking as Program Generation

Anselm Rothe\textsuperscript{1} \hspace{1cm} Brenden M. Lake\textsuperscript{1,2} \hspace{1cm} Todd M. Gureckis\textsuperscript{1}

\texttt{anselm@nyu.edu} \hspace{1cm} \texttt{brenden@nyu.edu} \hspace{1cm} \texttt{todd.gureckis@nyu.edu}

\textsuperscript{1}Department of Psychology \hspace{1cm} \textsuperscript{2}Center for Data Science
New York University

Abstract

A hallmark of human intelligence is the ability to ask rich, creative, and revealing questions. Here we introduce a cognitive model capable of constructing human-like questions. Our approach treats questions as formal programs that, when executed on the state of the world, output an answer. The model specifies a probability distribution over a complex, compositional space of programs, favoring concise programs that help the agent learn in the current context. We evaluate our approach by modeling the types of open-ended questions generated by humans who were attempting to learn about an ambiguous situation in a game. We find that our model predicts what questions people will ask, and can creatively produce novel questions that were not present in the training set. In addition, we compare a number of model variants, finding that both question informativeness and complexity are important for producing human-like questions.

1 Introduction

In active machine learning, a learner is able to query an oracle in order to obtain information that is expected to improve performance. Theoretical and empirical results show that active learning can speed acquisition for a variety of learning tasks [see 21, for a review]. Although impressive, most work on active machine learning has focused on relatively simple types of information requests (most often a request for a supervised label). In contrast, humans often learn by asking far richer questions which more directly target the critical parameters in a learning task. A human child might ask “Do all dogs have long tails?” or “What is the difference between cats and dogs?” [2]. A long term goal of artificial intelligence (AI) is to develop algorithms with a similar capacity to learn by asking rich questions. Our premise is that we can make progress toward this goal by better understanding human question asking abilities in computational terms [cf. 8].

To that end, in this paper, we propose a new computational framework that explains how people construct rich and interesting queries within a particular domain. A key insight is to model questions as programs that, when executed on the state of a possible world, output an answer. For example, a program corresponding to “Does John prefer coffee to tea?” would return \texttt{True} for all possible world states where this is the correct answer and \texttt{False} for all others. Other questions may return different types of answers. For example “How many sugars does John take in his coffee?” would return a number 0, 1, 2, etc. depending on the world state. Thinking of questions as syntactically well-formed programs recasts the problem of question asking as one of program synthesis. We show that this powerful formalism offers a new approach to modeling question asking in humans and may eventually enable more human-like question asking in machines.

We evaluate our model using a data set containing natural language questions asked by human participants in an information-search game [19]. Given an ambiguous situation or context, our model can predict what questions human learners will ask by capturing constraints in how humans construct semantically meaningful questions. The method successfully predicts the frequencies of
active learning for people and machines

rich human questions

How do they grow their babies?

Why is he up in the tree?

What is the difference between a shark and a fish?

simple machine questions

What is the category label of this object?

What is the category label of this object?

What is the category label of this object?
Experiment: Free-form question asking

(e.g., Markant & Gureckis, 2012, 2014)

Hidden configuration of ships

3 ships (blue, purple, red)
3 possible sizes (2-4 tiles)
1.6 million possible configurations
Experiment: Free-form question asking

(e.g., Markant & Gureckis, 2012, 2014)

Hidden configuration of ships

3 ships (blue, purple, red)
3 possible sizes (2-4 tiles)
1.6 million possible configurations

Phase 1: Sampling

Phase 2: Question asking

Is the red ship horizontal?

Constraints
• one word answers
• no combinations

Repeated for 18 different hidden configurations
**Location/standard queries**

- 24 What color is at [row][column]?
- 24 Is there a ship at [row][column]?
- 31 Is there a [color_incl_water] tile at [row][column]?

**Region queries**

- 4 Is there any ship in row [row]?
- 9 Is there any part of the [color] ship in row [row]?
- 5 How many tiles in row [row] are occupied by ships?
- 1 Are there any ships in the bottom half of the grid?
- 10 Is there any ship in column [column]?
- 10 Is there any part of the [color] ship in column [column]?
- 3 Are all parts of the [color] ship in column [column]?
- 2 How many tiles in column [column] are occupied by ships?
- 1 Is any part of the [color] ship in the left half of the grid?

**Ship size queries**

- 185 How many tiles is the [color] ship?
- 71 Is the [color] ship [size] tiles long?
- 8 Is the [color] ship [size] or more tiles long?
- 5 How many ships are [size] tiles long?
- 8 Are any ships [size] tiles long?
- 2 Are all ships [size] tiles long?
- 2 Are all ships the same size?
- 2 Do the [color1] ship and the [color2] ship have the same size?
- 3 Is the [color1] ship longer than the [color2] ship?
- 3 How many tiles are occupied by ships?

**Ship orientation queries**

- 94 Is the [color] ship horizontal?
- 7 How many ships are horizontal?
- 3 Are there more horizontal ships than vertical ships?
- 1 Are all ships horizontal?
- 4 Are all ships vertical?
- 7 Are the [color1] ship and the [color2] ship parallel?

**Adjacency queries**

- 12 Do the [color1] ship and the [color2] ship touch?
- 6 Are any of the ships touching?
- 9 Does the [color] ship touch any other ship?
- 2 Does the [color] ship touch both other ships?

**Demonstration queries**

- 14 What is the location of one [color] tile?
- 28 At what location is the top left part of the [color] ship?
- 5 At what location is the bottom right part of the [color] ship?
### Location/standard queries

<table>
<thead>
<tr>
<th>N</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>What color is at [row][column]?</td>
</tr>
<tr>
<td>24</td>
<td>Is there a ship at [row][column]?</td>
</tr>
<tr>
<td>31</td>
<td>Is there a [color_incl_water] tile at [row][column]?</td>
</tr>
</tbody>
</table>

| 10  | Is there any ship in column [column]?                                     |
| 10  | Is there any part of the [color] ship in column [column]?                 |
| 3   | Are all parts of the [color] ship in column [column]?                     |
| 2   | How many tiles in column [column] are occupied by ships?                  |
| 1   | Is any part of the [color] ship in the left half of the grid?             |

### Ship size queries

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<td>Is the [color1] ship longer than the [color2] ship?</td>
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<tr>
<td>3</td>
<td>How many tiles are occupied by ships?</td>
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### Ship orientation queries

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<td>94</td>
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</tr>
<tr>
<td>7</td>
<td>How many ships are horizontal?</td>
</tr>
<tr>
<td>3</td>
<td>Are there more horizontal ships than vertical ships?</td>
</tr>
<tr>
<td>1</td>
<td>Are all ships horizontal?</td>
</tr>
<tr>
<td>4</td>
<td>Are all ships vertical?</td>
</tr>
<tr>
<td>7</td>
<td>Are the [color1] ship and the [color2] ship parallel?</td>
</tr>
</tbody>
</table>

### Adjacency queries

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</table>

### Demonstration queries

<table>
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<tr>
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<th>Question</th>
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</thead>
<tbody>
<tr>
<td>14</td>
<td>What is the location of one [color] tile?</td>
</tr>
<tr>
<td>28</td>
<td>At what location is the top left part of the [color] ship?</td>
</tr>
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<td>5</td>
<td>At what location is the bottom right part of the [color] ship?</td>
</tr>
</tbody>
</table>
### Location/standard queries

| N   | What color is at [row][column]?
|-----|-------------------------------|
| 24  | Is there a ship at [row][column]?
| 31  | Is there a [color_incl_water] tile at [row][column]?

### Region queries

| 4   | Is there any ship in row [row]?
| 9   | Is there any part of the [color] ship in row [row]?
| 5   | How many tiles in row [row] are occupied by ships?
| 1   | Are there any ships in the bottom half of the grid?
| 10  | Is there any ship in column [column]?
| 10  | Is there any part of the [color] ship in column [column]?
| 3   | Are all parts of the [color] ship in column [column]?
| 2   | How many tiles in column [column] are occupied by ships?

### Ship size queries

<table>
<thead>
<tr>
<th>185</th>
<th>How many tiles is the [color] ship?</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td>Is the [color] ship [size] tiles long?</td>
</tr>
<tr>
<td>8</td>
<td>Is the [color] ship [size] or more tiles long?</td>
</tr>
<tr>
<td>5</td>
<td>How many ships are [size] tiles long?</td>
</tr>
<tr>
<td>8</td>
<td>Are any ships [size] tiles long?</td>
</tr>
<tr>
<td>2</td>
<td>Are all ships [size] tiles long?</td>
</tr>
</tbody>
</table>

### Adjacency queries

| 94  | Is the [color] ship horizontal? |
| 7   | How many ships are horizontal? |
| 3   | Are there more horizontal ships than vertical ships? |
| 1   | Are all ships horizontal? |
| 4   | Are all ships vertical? |
| 7   | Are the [color1] ship and the [color2] ship parallel? |

### Demonstration queries

| 12  | Do the [color1] ship and the [color2] ship touch? |
| 6   | Are any of the ships touching? |
| 9   | Does the [color] ship touch any other ship? |
| 2   | Does the [color] ship touch both other ships? |

### Other queries

<p>| 14  | What is the location of one [color] tile? |
| 28  | At what location is the top left part of the [color] ship? |
| 5   | At what location is the bottom right part of the [color] ship? |</p>
<table>
<thead>
<tr>
<th>N</th>
<th>Location/standard queries</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>What color is at [row][column]?</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Is there a ship at [row][column]?</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>Is there a [color_incl water] tile at [row][column]?</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Is there any ship in row [row]?</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Is there any part of the [color] ship in row [row]?</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>How many tiles in row [row] are occupied by ships?</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Are there any ships in the bottom half of the grid?</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Is there any ship in column [column]?</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Is there any part of the [color] ship in column [column]?</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Are all parts of the [color] ship in column [column]?</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>How many tiles in column [column] are occupied by ships?</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Is any part of the [color] ship in the left half of the grid?</td>
<td></td>
</tr>
<tr>
<td>185</td>
<td>How many tiles is the [color] ship?</td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>Is the [color] ship [size] tiles long?</td>
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<td>8</td>
<td>Is the [color] ship [size] or more tiles long?</td>
<td></td>
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<td>How many ships are [size] tiles long?</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Are any ships [size] tiles long?</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Are all ships [size] tiles long?</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Are all ships the same size?</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Do the [color1] ship and the [color2] ship have the same size?</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Is the [color1] ship longer than the [color2] ship?</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>How many tiles are occupied by ships?</td>
<td></td>
</tr>
<tr>
<td>94</td>
<td>Is the [color] ship horizontal?</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>How many ships are horizontal?</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Are there more horizontal ships than vertical ships?</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Are all ships horizontal?</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Are all ships vertical?</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Are the [color1] ship and the [color2] ship parallel?</td>
<td></td>
</tr>
</tbody>
</table>

**Ship orientation queries**

| 14 | Is the [color] ship horizontal?                                                             |  |
| 7  | How many ships are horizontal?                                                              |  |
| 3  | Are there more horizontal ships than vertical ships?                                        |  |
| 1  | Are all ships horizontal?                                                                   |  |
| 4  | Are all ships vertical?                                                                     |  |
| 7  | Are the [color1] ship and the [color2] ship parallel?                                       |  |

**Adjacency queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Demonstration queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Region queries**

| 4  | Is there any ship in row [row]?                                                            |  |
| 9  | Is there any part of the [color] ship in row [row]?                                        |  |
| 5  | How many tiles in row [row] are occupied by ships?                                         |  |
| 1  | Are there any ships in the bottom half of the grid?                                        |  |
| 10 | Is there any ship in column [column]?                                                      |  |
| 10 | Is there any part of the [color] ship in column [column]?                                   |  |
| 3  | Are all parts of the [color] ship in column [column]?                                      |  |
| 2  | How many tiles in column [column] are occupied by ships?                                    |  |
| 1  | Is any part of the [color] ship in the left half of the grid?                               |  |
| 185| How many tiles is the [color] ship?                                                        |  |
| 71 | Is the [color] ship [size] tiles long?                                                      |  |
| 8  | Is the [color] ship [size] or more tiles long?                                              |  |
| 5  | How many ships are [size] tiles long?                                                      |  |
| 8  | Are any ships [size] tiles long?                                                            |  |
| 2  | Are all ships [size] tiles long?                                                            |  |
| 2  | Are all ships the same size?                                                                |  |
| 2  | Do the [color1] ship and the [color2] ship have the same size?                              |  |
| 3  | Is the [color1] ship longer than the [color2] ship?                                         |  |
| 3  | How many tiles are occupied by ships?                                                       |  |

**Ship size queries**

| 185| How many tiles is the [color] ship?                                                        |  |
| 71 | Is the [color] ship [size] tiles long?                                                      |  |
| 8  | Is the [color] ship [size] or more tiles long?                                              |  |
| 5  | How many ships are [size] tiles long?                                                      |  |
| 8  | Are any ships [size] tiles long?                                                            |  |
| 2  | Are all ships [size] tiles long?                                                            |  |
| 2  | Are all ships the same size?                                                                |  |
| 2  | Do the [color1] ship and the [color2] ship have the same size?                              |  |
| 3  | Is the [color1] ship longer than the [color2] ship?                                         |  |
| 3  | How many tiles are occupied by ships?                                                       |  |

**Ship orientation queries**

| 94 | Is the [color] ship horizontal?                                                             |  |
| 7  | How many ships are horizontal?                                                              |  |
| 3  | Are there more horizontal ships than vertical ships?                                        |  |
| 1  | Are all ships horizontal?                                                                   |  |
| 4  | Are all ships vertical?                                                                    |  |
| 7  | Are the [color1] ship and the [color2] ship parallel?                                       |  |

**Adjacency queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Demonstration queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Ship size queries**

| 185| How many tiles is the [color] ship?                                                        |  |
| 71 | Is the [color] ship [size] tiles long?                                                      |  |
| 8  | Is the [color] ship [size] or more tiles long?                                              |  |
| 5  | How many ships are [size] tiles long?                                                      |  |
| 8  | Are any ships [size] tiles long?                                                            |  |
| 2  | Are all ships [size] tiles long?                                                            |  |
| 2  | Are all ships the same size?                                                                |  |
| 2  | Do the [color1] ship and the [color2] ship have the same size?                              |  |
| 3  | Is the [color1] ship longer than the [color2] ship?                                         |  |
| 3  | How many tiles are occupied by ships?                                                       |  |

**Ship orientation queries**

| 94 | Is the [color] ship horizontal?                                                             |  |
| 7  | How many ships are horizontal?                                                              |  |
| 3  | Are there more horizontal ships than vertical ships?                                        |  |
| 1  | Are all ships horizontal?                                                                   |  |
| 4  | Are all ships vertical?                                                                    |  |
| 7  | Are the [color1] ship and the [color2] ship parallel?                                       |  |

**Adjacency queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Demonstration queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |

**Ship size queries**

| 185| How many tiles is the [color] ship?                                                        |  |
| 71 | Is the [color] ship [size] tiles long?                                                      |  |
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| 5  | How many ships are [size] tiles long?                                                      |  |
| 8  | Are any ships [size] tiles long?                                                            |  |
| 2  | Are all ships [size] tiles long?                                                            |  |
| 2  | Are all ships the same size?                                                                |  |
| 2  | Do the [color1] ship and the [color2] ship have the same size?                              |  |
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**Ship orientation queries**

| 94 | Is the [color] ship horizontal?                                                             |  |
| 7  | How many ships are horizontal?                                                              |  |
| 3  | Are there more horizontal ships than vertical ships?                                        |  |
| 1  | Are all ships horizontal?                                                                   |  |
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| 7  | Are the [color1] ship and the [color2] ship parallel?                                       |  |

**Adjacency queries**

| 14 | What is the location of one [color] tile?                                                   |  |
| 28 | At what location is the top left part of the [color] ship?                                   |  |
| 5  | At what location is the bottom right part of the [color] ship?                               |  |
How do people think of a question to ask? question asking as program generation

Game primitives

“Size of the blue ship?” (size Blue)

“Color at tile A1?” (color A1)

“Orientation of the blue ship?” (orient Blue)

Primitive operators

(+ X X)

Novel questions

“What is the total size of all the ships?” (+

(+ (size Blue)

(size Red)

(size Purple))

Are the blue ship and the red ship parallel?” (=

(orient Blue)

(orient Red))
## Questions as programs

<table>
<thead>
<tr>
<th>GROUP</th>
<th>QUESTION</th>
<th>FUNCTION</th>
<th>EXPRESSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>location</td>
<td>What color is at A1?</td>
<td>location</td>
<td>(color A1)</td>
</tr>
<tr>
<td></td>
<td>Is there a ship at A1?</td>
<td>locationA</td>
<td>(not (= (color A1) Water))</td>
</tr>
<tr>
<td></td>
<td>Is there a blue tile at A1?</td>
<td>locationD</td>
<td>(= (color A1) Blue)</td>
</tr>
<tr>
<td>segmentation</td>
<td>Is there any ship in row 1?</td>
<td>row</td>
<td>(&gt; (+ (map (λ x (and (= (row x) 1) (not (= (color x) Water))) (set A1 ... F6))) 0)</td>
</tr>
<tr>
<td></td>
<td>Is there any part of the blue ship in row 1?</td>
<td>rowD</td>
<td>(&gt; (+ (map (λ x (and (= (row x) 1) (= (color x) Blue))) (set A1 ... F6))) 0)</td>
</tr>
<tr>
<td></td>
<td>Are all parts of the blue ship in row 1?</td>
<td>rowDL</td>
<td>(&gt; (+ (map (λ x (and (= (row x) 1) (= (color x) Blue))) (set A1 ... F6))) 1)</td>
</tr>
<tr>
<td></td>
<td>How many tiles in row 1 are occupied by ships?</td>
<td>rowNA</td>
<td>(+ (map (λ x (and (= (row x) 1) (not (= (color x) Water))) (set A1 ... F6)))</td>
</tr>
<tr>
<td></td>
<td>Are there any ships in the bottom half of the grid?</td>
<td>rowX2</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>Is there any ship in column 1?</td>
<td>col</td>
<td>(&gt; (+ (map (λ x (and (= (col x) 1) (not (= (color x) Water))) (set A1 ... F6))) 0)</td>
</tr>
<tr>
<td></td>
<td>Is there any part of the blue ship in column 1?</td>
<td>colD</td>
<td>(&gt; (+ (map (λ x (and (= (col x) 1) (= (color x) Blue))) (set A1 ... F6))) 0)</td>
</tr>
<tr>
<td></td>
<td>Are all parts of the blue ship in column 1?</td>
<td>colDL</td>
<td>(&gt; (+ (map (λ x (and (= (col x) 1) (= (color x) Blue))) (set A1 ... F6))) 1)</td>
</tr>
<tr>
<td></td>
<td>How many tiles in column 1 are occupied by ships?</td>
<td>colNA</td>
<td>(+ (map (λ x (and (= (col x) 1) (not (= (color x) Water))) (set A1 ... F6)))</td>
</tr>
<tr>
<td></td>
<td>Is any part of the blue ship in the left half of the grid?</td>
<td>colX1</td>
<td>...</td>
</tr>
<tr>
<td>ship size</td>
<td>How many tiles is the blue ship?</td>
<td>shipszie</td>
<td>(size Blue)</td>
</tr>
<tr>
<td></td>
<td>Is the blue ship 3 tiles long?</td>
<td>shipszieD</td>
<td>(= (size Blue) 3)</td>
</tr>
<tr>
<td></td>
<td>Is the blue ship 3 or more tiles long?</td>
<td>shipszieM</td>
<td>(or (= (size Blue) 3) (&gt; (size Blue) 3))</td>
</tr>
<tr>
<td></td>
<td>How many ships are 3 tiles long?</td>
<td>shipszieN</td>
<td>(+ (map (λ x (= (size x) 3)) (set Blue Red Purple)))</td>
</tr>
<tr>
<td></td>
<td>Are any ships 3 tiles long?</td>
<td>shipszieDA</td>
<td>(&gt; (+ (map (λ x (= (size x) 3)) (set Blue Red Purple))) 0)</td>
</tr>
<tr>
<td></td>
<td>Are all ships 3 tiles long?</td>
<td>shipszieDL</td>
<td>(= (+ (map (λ x (= (size x) 3)) (set Blue Red Purple))) 3)</td>
</tr>
<tr>
<td></td>
<td>Are all ships the same size?</td>
<td>shipszieL</td>
<td>(= (map (λ x (size x)) (set Blue Red Purple)))</td>
</tr>
<tr>
<td></td>
<td>Do the blue ship and the red ship have the same size?</td>
<td>shipszieX1</td>
<td>(= (size Blue) (size Red))</td>
</tr>
<tr>
<td></td>
<td>Is the blue ship longer than the red ship?</td>
<td>shipszieX2</td>
<td>(&gt; (size Blue) (size Red))</td>
</tr>
<tr>
<td>orientation</td>
<td>How many ships are horizontal?</td>
<td>totalshipsize</td>
<td>(+ (map (λ x (size x)) (set Blue Red Purple)))</td>
</tr>
<tr>
<td></td>
<td>Is the blue ship horizontal?</td>
<td>horizontal</td>
<td>(= (orient Blue) H)</td>
</tr>
<tr>
<td></td>
<td>How many ships are horizontal?</td>
<td>horizontalN</td>
<td>(+ (map (λ x (= (orient x) H) (set Blue Red Purple))))</td>
</tr>
<tr>
<td></td>
<td>Are there more horizontal ships than vertical ships?</td>
<td>horizontalM</td>
<td>(&gt; (+ (map (λ x (= (orient x) H) (set Blue Red Purple))) 1)</td>
</tr>
<tr>
<td></td>
<td>Are all ships horizontal?</td>
<td>horizontalL</td>
<td>(= (+ (map (λ x (= (orient x) H) (set Blue Red Purple))) 3)</td>
</tr>
<tr>
<td></td>
<td>Are all ships vertical?</td>
<td>verticalL</td>
<td>(= (+ (map (λ x (= (orient x) H) (set Blue Red Purple))) 0)</td>
</tr>
<tr>
<td>touching</td>
<td>Do the blue ship and the red ship touch?</td>
<td>touching</td>
<td>(touch Blue Red)</td>
</tr>
<tr>
<td></td>
<td>Are any of the ships touching?</td>
<td>touchingA</td>
<td>(or (touch Blue Red) (or (touch Blue Purple) (touch Red Purple)))</td>
</tr>
<tr>
<td></td>
<td>Does the blue ship touch any other ship?</td>
<td>touchingXA</td>
<td>(or (touch Blue Red) (touch Blue Purple))</td>
</tr>
<tr>
<td></td>
<td>Does the blue ship touch both other ships?</td>
<td>touchingX1</td>
<td>(and (touch Blue Red) (touch Blue Purple))</td>
</tr>
<tr>
<td>demonstration</td>
<td>What is the location of one blue tile?</td>
<td>demonstration</td>
<td>(draw (select (set A1 ... F6) Blue))*</td>
</tr>
<tr>
<td></td>
<td>At what location is the top left part of the blue ship?</td>
<td>topleft</td>
<td>(topleft Blue)</td>
</tr>
<tr>
<td></td>
<td>At what location is the bottom right part of the blue ship?</td>
<td>bottomright</td>
<td>(bottomright Blue)</td>
</tr>
</tbody>
</table>
We need a task that provides both, (a) flexibility to facilitate constraints to allow on a broader level: How do people generate questions? How do you tell if a question is “good”? How well do people perform when being allowed to naturally generate rich questions? However, people use a much richer set of questions to obtain information in everyday questions, called.

Measures

Expected Savings (ES)

Simulate for each possible answer: how much would this answer reduce your uncertainty about which gameboard configuration.

Active Learning with Rich Queries

Asking Useful Questions:

Start Here!

Department of Psychology, New York University

Asking Useful Questions: What is this? Shannon entropy is transformed, by applying transformation to all questions is that they will return

Identical to Experiment 1, but instead of generating questions participants were presented with 6 questions gene

Three representative example trials)

Available questions are given.

One might do well to be skeptical about claims of optimal or rational inquiry in cases where the set of

A bottleneck for active inquiry is first generating the relevant question. After that, evaluation is pretty easy.

After getting familiar with the game, participants generated questions for 18 Battleship gameboards.

Experiment 1

Free Play Phase

Trial 4

Phase 2: Painting

5 Trials

18 Trials

40 participants on Amazon Mechanical Turk

15

10

15

Phase 2: Ship Indication

Questions generated by the model with the highest expected information gain are compared to the model with the lower expected information gain. The objectively best questions and the model with the highest expected information gain are used to evaluate the model.

Selected References


What is this?

Answer

Is there a ship at 2B?

Is the red ship 3 tiles long?

What is the location of one purple tile?

At what location is the top left part of the red ship?

Is the red ship horizontal?

Is the red ship 2 tiles long?

At what location is the top left part of the purple ship?

Is there a ship at 2B?

Is the red ship horizontal?

Is the red ship 2 tiles long?

At what location is the top left part of the red ship?

Is the red ship horizontal?

Is the red ship 2 tiles long?

At what location is the top left part of the purple ship?

Is there a ship at 2B?

Is the red ship horizontal?

Is there a red tile at A1?

Is there a blue tile at A1?
Learning a probabilistic generative model of questions

Goal: predict human questions in novel scenarios

\[ \mathcal{X} : \text{question / program} \]
\[ f(\cdot) : \text{features (Expected Info. Gain, length, answer type, etc.)} \]
\[ \theta : \text{trainable parameters} \]

energy:

\[ \mathcal{E}(x) = \theta_1 f_1(x) + \theta_2 f_2(x) + \cdots + \theta_K f_K(x) \]

generative model:

\[ P(x; \theta) = \frac{\exp^{-\mathcal{E}(x)}}{\sum_{x' \in \mathcal{X}} \exp^{-\mathcal{E}(x')}} \]

maximum likelihood fitting:

\[ \arg\max_\theta P(D; \theta) \]

\[ D : \text{empirical data} \]
Asking novel questions through program generation

### Context 7

<table>
<thead>
<tr>
<th>EIG</th>
<th>Question/Program</th>
<th>Energy</th>
<th>Human</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.44</td>
<td>How many tiles are occupied by ships? (++ (map (lambda x (size x)) (set Blue Red Purple)))</td>
<td>6.53</td>
<td></td>
</tr>
<tr>
<td>1.79</td>
<td>How many ships are 4 tiles long? (++ (map (lambda x (== (size x) 4)) (set Blue Red Purple)))</td>
<td>7.88</td>
<td></td>
</tr>
<tr>
<td>8.90</td>
<td>Are all the ships horizontal? (all (map (lambda x (== H (orient x))) (set Blue Red Purple)))</td>
<td>10.51</td>
<td></td>
</tr>
<tr>
<td>12.89</td>
<td>Are any of the ship sizes greater than 2? (any (map (lambda x (&gt; (size x) 2)) (set Blue Red Purple)))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Context 9

<table>
<thead>
<tr>
<th>EIG</th>
<th>Question/Program</th>
<th>Energy</th>
<th>Human</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.59</td>
<td>How many tiles in row 4 are occupied by ships? (++ (map (lambda y (and (== (rowL y) 4) (not (== (color y) Water))) (set 1A ... 6F)))</td>
<td>7.48</td>
<td></td>
</tr>
<tr>
<td>1.56</td>
<td>How many tiles is the purple ship? (size Purple)</td>
<td>8.74</td>
<td></td>
</tr>
<tr>
<td>9.94</td>
<td>What is the top left of all the ship tiles? (topleft (setDifference (set 1A ... 6F) (coloredTiles Water)))</td>
<td>10.98</td>
<td></td>
</tr>
<tr>
<td>16.34</td>
<td>Are blue and purple ships touching and red and purple not touching (or vice versa)? (== (touch Blue Purple) (not (touch Red Purple)))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>